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The thermohaline Rayleigh–Jeffreys problem

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The onset of convection induced by thermal and solute concentration gradients, in a horizontal layer of a viscous fluid, is studied by means of linear stability analysis. A Fourier series method is used to obtain the eigenvalue equation, which involves a thermal Rayleigh number R and an analogous solute Rayleigh number S, for a general set of boundary conditions. Numerical solutions are obtained for selected cases. Both oscillatory and monotonic instability are considered, but only the latter is treated in detail. The former can occur when a strongly stabilizing solvent gradient is opposed by a destablizing thermal gradient. When the same boundary equations are required to be satisfied by the temperature and concentration perturbations, the monotonic stability boundary curve in the (R, S)-plane is a straight line. Otherwise this curve is concave towards the origin. For certain combinations of boundary conditions the critical value of R does not depend on S (for some range of S) or vice versa. This situation pertains when the critical horizontal wave-number is zero.

A general discussion of the possibility and significance of convection at 'zero' wave-number (single convection cell) is presented in an appendix.

1. Introduction

The problem of the onset of convection induced by buoyancy effects due to vertical thermal gradients, in a horizontal layer of a viscous fluid, was first analysed by Rayleigh (1916) and Jeffreys (1926, 1928). Because of the historical importance of the experiments of Bénard (1900, 1901), this problem is sometimes referred to as the Bénard problem. It is now known (see, for example, Pearson (1958) or a review by Nield (1965)) that surface tension effects, and not buoyancy effects, were dominant in Bénard's experiments.

Buoyancy forces can arise not only from density differences due to variations in temperature but also from those due to variations in solute concentration. In problems of interest the solute is commonly, but not necessarily, a salt. Such problems arise in oceanography, limnology and engineering. Examples of particular interest are provided by some Antarctic lakes (Shirtcliffe 1964; Hoare 1966) and ponds built to trap solar heat (Tabor & Matz 1965).

When the density of a stratified layer of a single-component fluid decreases upwards, the configuration is stable. This is not necessarily so for a fluid consisting of two or more components which can diffuse relative to each other. The reason lies in the fact that the diffusivity of heat is usually much greater than the diffusivity of a solute. A displaced particle of fluid thus loses any excess heat more rapidly than any excess solute. The resulting buoyancy force may tend to increase the displacement of the particle from its original position, and thus cause instability. The same effect may also cause overstability (involving oscillatory motions of increasing amplitude). We can see this by considering a fluid whose density decreases with increase of temperature and increases with concentration of dissolved salt. Suppose that the gradients of temperature and concentration each decrease upwards. A particle of fluid displaced upwards is initially warmer and saltier than its surroundings. Its temperature rapidly declines to that of its surroundings, but its salt concentration declines slowly. Because of the excess salt content, it may become more dense than its surroundings. The buoyancy force then acts to restore it to its original position. On arrival there it still has most of its original salt content, but because it has spent some time in a cooler region it may still be more dense than its surroundings. The buoyancy force then causes the particle to overshoot its original position. Oscillations of increasing amplitude therefore result.

A study of the onset of convection in a layer of sugar solution, with a stabilizing concentration gradient, when the layer is heated from below, has been made by Shirtcliffe (1967). He has found that the first stage of the development of convection layers similar to those described by Turner & Stommel (1964) is the appearance in a thin bottom layer of a cellular oscillatory motion which initially has a very definite period.

Vertgeim (1955) gave a theoretical treatment of the onset of monotonic thermohaline convection in a vertical cylinder, while both monotonic and oscillatory convection were considered by Gershuni & Zhukhovitskii (1963) for fluid between two parallel vertical plates. For a horizontal layer of fluid, Stern (1960) discussed monotonic instability, and Lieber & Rintel (1963) considered the possibility of overstability. The boundary conditions treated in each of these papers have been the mathematically convenient ones only. Thus for a horizontal layer both top and bottom boundaries were assumed to be free, with the temperature and concentration there kept fixed. (We shall refer to these as 'ideal' boundaries.) Although Walin (1964) ignored the boundary conditions in his treatment of an unbounded fluid, his analysis implicitly depends on the assumption of ideal boundaries since he considered disturbances involving one Fourier component only. Thus his results are appropriate to a horizontal layer, of very large depth, between ideal boundaries. Similar results were obtained by Weinberger (1962, 1964). In the present paper more-realistic boundaries (at least one rigid) are considered, and general conditions for the temperature and concentration are applied. We shall find that although the thermal and solute effects are perfectly coupled for the case of ideal boundaries, this is not always so for other cases.

The present study is confined to infinitesimal disturbances. When the solute gradient is stabilizing, Veronis (1965) and Sani (1965) have found that finite-amplitude subcritical instability (convection at a thermal Rayleigh number less than that given by the linear theory) is possible. The apparent reason for this is that non-linear terms in the governing equations may lead to a substantial distortion of the previously linear solute profile, so that the solute gradient in the bulk of the fluid is reduced. The distortion of the thermal profile is less because of the relatively high thermal diffusivity. Away from the horizontal boundaries this destablizing thermal gradient may then become dominant so that convection thus ensues. If the basic solute gradient is destablizing then any finite distortion tends to reduce its destablizing effect. An infinitesimal disturbance should then be the most unstable one, and the theory below therefore applicable.

The requisite perturbation analysis is given in § 2. Since the full solution for the case of ideal boundary conditions has not yet appeared in the general literature, it is included here in § 3. In § 4 the Fourier series method is used to obtain the eigenvalue equation for general boundary conditions. Solutions of this equation are presented and discussed in § 5. Consideration is required of the situation when the critical horizontal wave-number is zero. This situation is discussed in detail in the appendix. Among the features of interest is the ease with which the critical Rayleigh number may then be found.

2. Perturbation analysis

The Boussinesq approximation for a quasi-incompressible fluid enables one, following Chandrasekar (1961) and Yih (1965), to write the governing equations as

$$\nabla.\mathbf{u} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{\rho}{\rho_0}g\mathbf{k} - \frac{1}{\rho_0}\nabla P + \nu\nabla^2\mathbf{u}, \tag{2.2}$$

$$\partial T/\partial t + \mathbf{u}.\nabla T = \kappa\nabla^2 T, \tag{2.3}$$

$$\partial C/\partial t + \mathbf{u}.\nabla C = \kappa'\nabla^2 C, \tag{2.4}$$

$$\rho = \rho_0[1 - \alpha(T - T_0) - \alpha'(C - C_0)]. \tag{2.5}$$

Cartesian co-ordinates have been taken with the origin in the lower boundary and the z-axis vertically upwards (so that the fluid lies between the planes $z = 0$ and $z = d$). The dependent variables are the velocity $\mathbf{u} = (u, v, w)$, density ρ, pressure P, temperature T and solute mass concentration C. The gravitational acceleration is denoted by g, and a unit vector in the z-direction by $\mathbf{k}$. The kinematic viscosity ν, the thermal diffusivity κ and the solute diffusivity κ' are each assumed to be constant. Equations (2.1)–(2.4) express the conservation of total mass, momentum, heat and solute. The equation of state (2.5) involves a thermal coefficient of expansion α and an analogous solvent coefficient α'. The suffix zero refers to values at the reference level $z = 0$.

We suppose that the temperature and concentration at the lower and upper boundary have the uniform values T_0, C_0 and T_1, C_1 respectively. The steady state solution is

$$\mathbf{u} = 0, \quad T_* = T_0 - \beta z, \quad C_* = C_0 - \beta' z,$$

where $\beta = (T_0 - T_1)/d$ and $\beta' = (C_0 - C_1)/d$ are the magnitudes of the uniform temperature and concentration gradients (positive if the quantities decrease upwards).

We now consider a perturbation on the steady state solution, and write

$$T = T_* + \theta, \quad C = C_* + \gamma, \quad P = P_* + p.$$

Then equations (2.1)–(2.4) give, to first order,

$$\nabla . \mathbf{u} = 0, \tag{2.6}$$

$$\frac{\partial \mathbf{u}}{\partial t} = g(\alpha\theta + \alpha'\gamma)\,\mathbf{k} - \nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\mathbf{u}, \tag{2.7}$$

$$\partial\theta/\partial t = \beta w + \kappa\nabla^2\theta, \tag{2.8}$$

$$\partial\gamma/\partial t = \beta' w + \kappa'\nabla^2\gamma. \tag{2.9}$$

We obtain, by manipulating (2.6) and (2.7),

$$(\partial/\partial t)\,(\nabla^2 w) = g\nabla_1^2(\alpha\theta + \alpha'\gamma) + \nu\nabla^4 w, \tag{2.10}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$.

The horizontal boundaries may be either rigid or free. For a rigid boundary the no-slip condition and the equation of continuity lead to the conditions $w = \partial w/\partial z = 0$. At a plane boundary where the tangential stress is zero the corresponding conditions are $w = \partial^2 w/\partial z^2 = 0$. There is no argument about the conditions appropriate to a rigid boundary, but Yih (1965) has pointed out that the above conditions (which have been used by many previous authors) for a free boundary are not strictly correct if deformation of the boundary is considered. The full conditions are considerably more involved. At present, when there is a paucity of experimental results, the effort required to treat the full conditions does not appear to be justified, and we shall adopt the simplified conditions. No error is involved for monotonic instability, and for oscillatory instability the approximation is acceptable provided the fluid layer is not too thin.

If the temperature at a boundary is kept fixed, then there $\theta = 0$. On the other hand, if the heat flux across the boundary is kept fixed (and the perturbation heat flux is thus zero) then $\partial\theta/\partial z = 0$ at the boundary. More generally, we may apply a 'radiation' type condition, $\partial\theta/\partial z + L\theta = 0$, where the sign of the parameter L must be chosen to ensure that the perturbation heat transfer is out of the fluid layer. We impose on γ a similar condition, $\partial\gamma/\partial z + M\gamma = 0$. If the boundary is impermeable then $M = 0$, while if the concentration is kept constant (for example, at its saturated value) then $M = \infty$ gives the appropriate condition.

Thus our boundary conditions will be

$$w = 0, \quad \frac{\partial w}{\partial z} - K_l d\frac{\partial^2 w}{\partial z^2} = 0, \quad d\frac{\partial\theta}{\partial z} - L_l\theta = 0, \quad d\frac{\partial\gamma}{\partial z} - M_l\gamma = 0, \quad \text{at} \quad z = 0, \tag{2.11}$$

$$w = 0, \quad \frac{\partial w}{\partial z} + K_u d\frac{\partial^2 w}{\partial z^2} = 0, \quad d\frac{\partial\theta}{\partial z} + L_u\theta = 0, \quad d\frac{\partial\gamma}{\partial z} + M_u\gamma = 0, \quad \text{at} \quad z = d, \tag{2.12}$$

where the suffices l, u refer to the lower and upper boundaries respectively, and for convenience we have introduced a parameter K which takes the discrete values 0 (for a rigid boundary) and ∞ (for a free boundary).

We now make a normal mode expansion and introduce non-dimensional variables. We let

$$[\theta, \gamma, w] = [\Theta(z), \Gamma(z), W(z)]\exp\{\sigma t + i(k_x x + k_y y)\}. \tag{2.13}$$

Here σ is a time constant (a complex number in general) and the horizontal wave-number is given by $k = \sqrt{(k_x^2 + k_y^2)}$. We take d/π as unit of length and write

$D = d/dz$ with z now expressed in terms of this new unit. We also let $W_1 = Wd/\pi\nu$, $\Theta_1 = \Theta\pi\kappa/\beta d\nu$, $\Gamma_1 = \Gamma\pi\kappa'/\beta' d\nu$ and $\sigma_1 = \sigma d^2/\pi^2\nu$. Equations (2.8) (2.9) and (2.10) then give

$$(D^2-b^2-r\sigma_1)\,\Theta_1 = -W_1, \tag{2.14}$$

$$(D^2-b^2-s\sigma_1)\,\Gamma_1 = -W_1, \tag{2.15}$$

and

$$(D^2-b^2)\,(D^2-b^2-\sigma_1)\,W_1 = Rb^2\Theta_1+Sb^2\Gamma_1, \tag{2.16}$$

where $\pi^4R = \alpha\beta g d^4/\kappa\nu$ is the thermal Rayleigh number and $\pi^4S = \alpha'\beta' g d^4/\kappa'\nu$ is the analogous solute Rayleigh number, while $r = \nu/\kappa$ is the Prandtl number and $s = \nu/\kappa'$ is the Schmidt number.

The boundary conditions (2.11) and (2.12) now become

$$W_1 = 0,\quad [D-\pi K_l D^2]\,W_1 = 0,\quad [D-L_l/\pi]\,\Theta_1 = 0,\quad [D-M_l/\pi]\,\Gamma_1 = 0,\quad \text{at}\quad z = 0, \tag{2.17}$$

and

$$W_1 = 0,\quad [D+\pi K_u D^2]\,W_1 = 0,\quad [D+L_u/\pi]\,\Theta_1 = 0,\quad [D+M_u/\pi]\,\Gamma_1 = 0,\quad \text{at}\quad z = \pi. \tag{2.18}$$

The differential equations (2.14), (2.15), (2.16) and the boundary conditions (2.17), (2.18), form an eigenvalue system of the eighth order.

3. Solution for 'ideal' boundary conditions

For the case when both boundaries are free, at fixed temperature and with fixed concentration, the boundary conditions are

$$W_1 = D^2W_1 = \Theta_1 = \Gamma_1 = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = \pi.$$

These conditions, and the differential equations (2.14)–(2.16), are satisfied by

$$[W_1, \Theta_1, \Gamma_1] = [1, (n^2+b^2+r\sigma_1)^{-1}, (n^2+b^2+s\sigma_1)^{-1}]\sin nz,$$

if

$$(n^2+b^2)\,(n^2+b^2+\sigma_1)\,(n^2+b^2+r\sigma_1)\,(n^2+b^2+s\sigma_1) = Rb^2(n^2+b^2+s\sigma_1)+Sb^2(n^2+b^2+r\sigma_1). \tag{3.1}$$

At neutral stability the real part of the time-constant σ_1 is zero. We thus put $\sigma_1 = i\omega$, where ω is real. The real and imaginary parts of this equation then give

$$(n^2+b^2)\,\{(n^2+b^2)^3-(R+S)\,b^2-\omega^2(rs+r+s)\,(n^2+b^2)\} = 0, \tag{3.2}$$

$$\omega\{(r+s+1)\,(n^2+b^2)^3-(sR+rS)\,b^3-\omega^2rs(n^2+b^2)\} = 0. \tag{3.3}$$

If $\omega = 0$, then

$$R+S = (n^2+b^2)^3/b^2, \tag{3.4}$$

while if $\omega \neq 0$, then

$$rs(n^2+b^2)\,\omega^2 = (r+s+1)\,(n^2+b^2)^3-(sR+rS)\,b^2, \tag{3.5}$$

and

$$\frac{s^2R}{(r+s)\,(s+1)}+\frac{r^2S}{(r+s)\,(r+1)} = \frac{(n^2+b^2)^3}{b^2}. \tag{3.6}$$

For various integers n the quantity $(n^2+b^2)^3/b^2$ is least when $n = 1$. The minimum value as b varies is then $\frac{27}{4}$, which is attained when $b^2 = \frac{1}{2}$. Hence the neutral stability loci in the (R, S)-plane are the straight lines

$$R+S = \tfrac{27}{4}, \tag{3.7}$$

and

$$\frac{s^2R}{(r+s)\,(s+1)}+\frac{r^2S}{(r+s)\,(r+1)} = \frac{27}{4}. \tag{3.8}$$

The first line corresponds to monotonic or stationary instability and the second to oscillatory instability or overstability (provided that ω, as given by equation (3.5), is real, i.e. $(sR+rS)/(r+s+1) \leqslant \frac{27}{4}$.)

If $r = s$ (that is $\kappa = \kappa'$) these lines are parallel. The second is further from the origin, and the condition for stability is $R+S < \frac{27}{4}$. Overstability is then ruled out. Except for this exceptional case, the lines intersect at the point

$$R = 6{\cdot}75(s+1)/(s-r), \quad S = 6{\cdot}75(r+1)/(r-s).$$

At this point the overstability curve bifurcates from the monotonic instability curve. Except for the linear magnification factor $\frac{27}{4}$, the stability diagram is similar to that given by Gershuni & Zhukhovitskii (1963) for a fluid between parallel vertical walls. The author's (1966*b*) thesis contains the results of a detailed examination, in the manner of Weiss (1964), of the nature of the roots of the equation (3.1) in σ_1. These results confirm that if

$$R+S < \tfrac{27}{4} \quad \text{and} \quad s^2(s+1)^{-1}R + r^2(r+1)^{-1}S < \tfrac{27}{4}(r+s)$$

then convection does not occur. When R and S are varied so that the first inequality (but not the second) is reversed then monotonic convection results, but if the second inequality only is reversed then oscillatory convection results. It is noteworthy that a negative density gradient $(sR+rS < 0)$ is not a sufficient condition for stability for either monotonic disturbances or oscillatory disturbances.

4. Solution for general boundary conditions

We now return to the general case where the boundary conditions are given by (2.17) and (2.18). The Fourier series method presented by Nield (1964) is very convenient for the present problem. The method involves expanding each dependent variable in two ways: first, as a Fourier series which can be differentiated the required number of times and, secondly, in an equivalent form suitable for imposing the boundary conditions. The second form is the sum of a simple polynomial and a Fourier series.

We denote the constants to be eliminated by

$$\lambda_1 = D^2W_1(0),\ \lambda_2 = D^2W_1(\pi),\ \lambda_3 = \Theta_1(0),\ \lambda_4 = \Theta_1(\pi),\ \lambda_5 = \Gamma_1(0),\ \lambda_6 = \Gamma_1(\pi),$$

and let

$$W_1 = \Sigma\{a_n - (2/\pi n^3)[\lambda_1 - (-1)^n\lambda_2]\}\sin nz \tag{4.1a}$$

$$= \Sigma a_n \sin nz - (1/6\pi)[\lambda_1 z(z-\pi)(z-2\pi) - \lambda_2 z(z^2-\pi^2)]. \tag{4.1b}$$

(In these sums, and those following, n runs from 1 to ∞.) Then, since W_1 vanishes at $z = 0$ and $z = \pi$, from the boundary conditions (2.17) and (2.18),

$$DW_1 = \Sigma n a_n \cos nz - (6\pi)^{-1}[\lambda_1(3z^2 - 6\pi z + 2\pi^2) - \lambda_2(3z^2-\pi^2)], \tag{4.2}$$

$$D^2W_1 = \Sigma\{-n^2a_n + (2/\pi n)[\lambda_1 - (-1)^n\lambda_2]\}\sin nz \tag{4.3a}$$

$$= \Sigma\{-n^2a_n\}\sin nz - \pi^{-1}[\lambda_1(z-\pi) - \lambda_2 z], \tag{4.3b}$$

$$D^4W_1 = \Sigma n^4 a_n \sin nz. \tag{4.4}$$

We also let
$$\Theta_1 = \Sigma\{b_n + (2/\pi n)\,[\lambda_3 - (-1)^n \lambda_4]\}\sin nz \tag{4.5a}$$
$$= \Sigma b_n \sin nz - \pi^{-1}[\lambda_3(z-\pi) - \lambda_4 z], \tag{4.5b}$$
so that
$$D\Theta_1 = \Sigma n b_n \cos nz - \pi^{-1}[\lambda_3 - \lambda_4], \tag{4.6}$$
$$D^2\Theta_1 = \Sigma(-n^2 b_n)\sin nz. \tag{4.7}$$

Finally, Γ_1 and its derivatives are given by similar expressions with c_n, λ_5, λ_6 replacing b_n, λ_3, λ_4.

The remaining boundary conditions require that
$$\Sigma \pi n a_n = (\tfrac{1}{3}\pi^2 + K_l)\,\lambda_1 + \tfrac{1}{6}\pi^2\lambda_2, \tag{4.8}$$
$$\Sigma(-1)^n \pi n a_n = -\tfrac{1}{6}\pi^2\lambda_1 - (\tfrac{1}{3}\pi^2 + K_u)\,\lambda_2, \tag{4.9}$$
$$\Sigma \pi n b_n = (1 + L_l)\,\lambda_3 - \lambda_4, \tag{4.10}$$
$$\Sigma(-1)^n \pi n b_n = \lambda_3 - (1 + L_u)\,\lambda_4, \tag{4.11}$$
$$\Sigma \pi n c_n = (1 + M_l)\,\lambda_5 - \lambda_6, \tag{4.12}$$
$$\Sigma(-1)^n \pi n c_n = \lambda_5 - (1 + M_u)\,\lambda_6. \tag{4.13}$$

The differential equations (2.14)–(2.16) are satisfied by substituting the complete Fourier expansions for W_1, Θ_1, Γ_1 and their derivatives of even order, and equating the coefficients of $\sin nz$. We thus obtain, with (n^2+b^2) denoted by F,
$$\begin{aligned} F(F+\sigma_1)\,a_n - Rb^2 b_n - Sb^2 c_n &= (2/\pi n)\{[2b^2 + \sigma_1 + n^{-2}(b^4 + \sigma_1 b^2)]\,[\lambda_1 - (-1)^n \lambda_2] \\ &\quad + Rb^2[\lambda_3 - (-1)^n \lambda_4] + Sb^2[\lambda_5 - (-1)^n \lambda_6]\}, \\ -a_n + (F + r\sigma_1)\,b_n &= (2/\pi n)\{n^{-2}[\lambda_1 - (-1)^n \lambda_2] + (b^2 + r\sigma_1)\,[\lambda_3 - (-1)^n \lambda_4]\}, \\ -a_n + (F + s\sigma_1)\,c_n &= (2/\pi n)\{n^{-2}[\lambda_1 - (-1)^n \lambda_2] + (b^2 + s\sigma_1)\,[\lambda_5 - (-1)^n \lambda_6]\}. \end{aligned}$$

From these equations a_n, b_n, c_n can be expressed in terms of $\lambda_1, \ldots, \lambda_6$. Substitution in (4.8)–(4.13) then yields six equations in $\lambda_1, \ldots, \lambda_6$. Elimination of these constants gives the eigenvalue equation.

For the general case this equation involves a lengthy expression. We are concerned mainly with the onset of steady convection, for which $\sigma_1 = 0$. When we write
$$\begin{aligned} f &= \pi b \coth \pi b, & g &= \pi b \,\mathrm{cosech}\, \pi b, \\ F &= n^2 + b^2, & G &= F(F^3 - Rb^2 - Sb^2), \\ \Sigma_1 &= \Sigma 2n^2F^2/G, & \Sigma_2 &= \Sigma(-1)^n\, 2n^2F^2/G, \\ \Sigma_3 &= \Sigma 2n^2b^2/G, & \Sigma_4 &= \Sigma(-1)^n\, 2n^2b^2/G, \\ \Sigma_5 &= \Sigma 2n^2b^2F/G, & \Sigma_6 &= \Sigma(-1)^n\, 2n^2b^2F/G, \end{aligned}$$
where the sums are all from $n = 1$ to $n = \infty$, the eigenvalue equation for this case is
$$\begin{vmatrix} \Sigma_1 + K_l & \Sigma_2 & R\Sigma_5 & R\Sigma_6 & S\Sigma_5 & S\Sigma_6 \\ \Sigma_2 & \Sigma_1 + K_u & R\Sigma_6 & R\Sigma_5 & S\Sigma_6 & S\Sigma_5 \\ \Sigma_5/b^2 & \Sigma_6/b^2 & R\Sigma_3 - f - L_l & R\Sigma_4 - g & S\Sigma_3 & S\Sigma_4 \\ \Sigma_6/b^2 & \Sigma_5/b^2 & R\Sigma_4 - g & R\Sigma_3 - f - L_u & S\Sigma_4 & S\Sigma_3 \\ \Sigma_5/b^2 & \Sigma_6/b^2 & R\Sigma_3 & R\Sigma_4 & S\Sigma_3 - f - M_l & S\Sigma_4 - g \\ \Sigma_6/b^2 & \Sigma_5/b^2 & R\Sigma_4 & R\Sigma_3 & S\Sigma_4 - g & S\Sigma_3 - f - M_u \end{vmatrix} = 0. \tag{4.14}$$

5. Results and discussion

In general the eigenvalue equation (4.14) must be solved numerically. If desired, the speed of convergence of each series may be increased in the usual way. For some boundary conditions, however, known results make a numerical calculation unnecessary. If $L_l = L_u$ and $M_l = M_u$ then the determinant in (4.14) reduces to the product of the determinants

$$\begin{vmatrix} f+L_l & g \\ g & f+L_u \end{vmatrix}$$

and

$$\begin{vmatrix} \Sigma_1+K_l & \Sigma_2 & (R+S)\Sigma_5 & (R+S)\Sigma_6 \\ \Sigma_2 & \Sigma_1+K_u & (R+S)\Sigma_6 & (R+S)\Sigma_5 \\ \Sigma_5/b^2 & \Sigma_6/b^2 & (R+S)\Sigma_3-f-L_l & (R+S)\Sigma_4-g \\ \Sigma_6/b^2 & \Sigma_5/b^2 & (R+S)\Sigma_4-g & (R+S)\Sigma_3-f-L_u \end{vmatrix}$$

(This follows when rows 3 and 4 are subtracted from rows 5 and 6 respectively, columns 3 and 4 are added to columns 5 and 6 respectively, and a Laplace expansion is made according to the second-order minors of the last two rows.) It is evident that, in this case, R and S enter the eigenvalue equation in the combination $(R+S)$ only. The problem then reduces to the usual thermal case with R replaced by $(R+S)$. Thus when the temperature perturbation θ and the concentration perturbation γ satisfy formally identical boundary conditions, the stability boundary curve in the (R, S)-plane is the straight line

$$R+S = R_c\,(=S_c),$$

where R_c is the critical value of R when $S = 0$, and S_c is that of S when $R = 0$. The critical wave-number b_c is the same for all (R, S) combinations. The flow pattern in cells induced by thermal effects is identical with that in cells induced by solute effects, or by a combination of these.

For convenience we have listed in table 1 critical values of the thermal Rayleigh number $R_c^* = \pi^4 R_c$ and the corresponding wave-number $a_c = \pi b_c$ for various limiting values of K_l, K_u, L_l and L_u. The same table, with S_c^*, M_l and M_u replacing R_c^*, L_l and L_u respectively, gives the critical solute Rayleigh number and the corresponding critical wave-number in the absence of the thermal effects.

When the boundary conditions on θ differ from those on γ, the stability boundary is no longer a straight line but is concave towards the origin. This is because the flow pattern in cells induced by thermal effects alone is now different from that in cells induced by solute effects alone. Considerations of energy balance require that $R/R_c + S/S_c \geqslant 1$. The argument is identical to that given in the author's papers on coupled surface-tension and buoyancy effects in this problem (Nield 1964, 1966*a*).

This is well illustrated in figures 1 and 2, where the stability boundary in the (R, S)-plane, and the critical wave-number corresponding to a point on this boundary, are plotted for two cases of possible practical interest. In each case the lower boundary is assumed to be rigid ($K_l = 0$) and the upper boundary to be free ($K_u = \infty$), and each is at fixed temperature ($L_l = \infty$, $L_u = \infty$); also the concentration gradient at the upper surface is in each case kept fixed ($M_u = 0$). In case (*a*) the concentration gradient at the lower boundary is kept fixed ($M_l = 0$), while in case (*b*) the concentration itself is kept fixed ($M_l = \infty$). Case (*a*) would be a good approximation to the conditions for a liquid layer bounded by metal below and air above. Case (*b*) would be applicable if additional undissolved solute present at the bottom kept the concentration there at its saturated value.

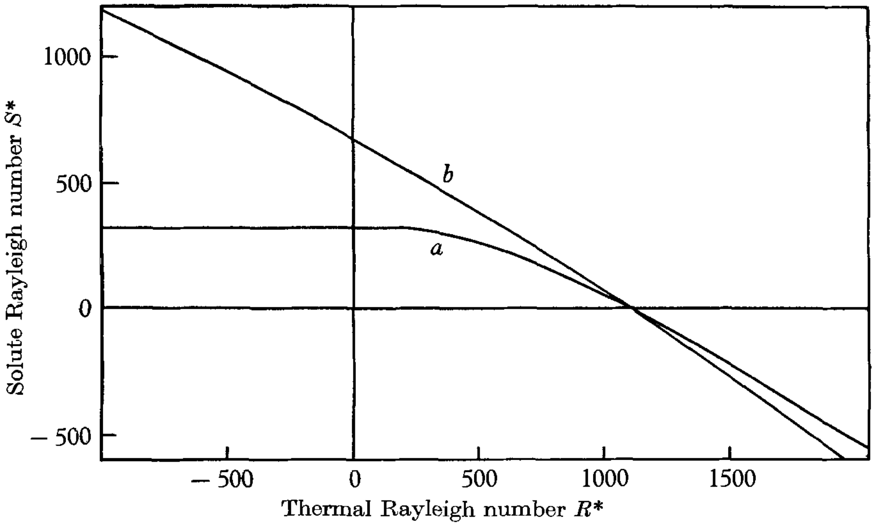

FIGURE 1. Stability diagram for infinitesimal monotonic disturbances. Stable states are represented by points below the appropriate boundary curve. The cases shown correspond to an upper surface which is free, conducting and impermeable, and a lower surface which is rigid, conducting and (*a*) impermeable or (*b*) at fixed concentration.

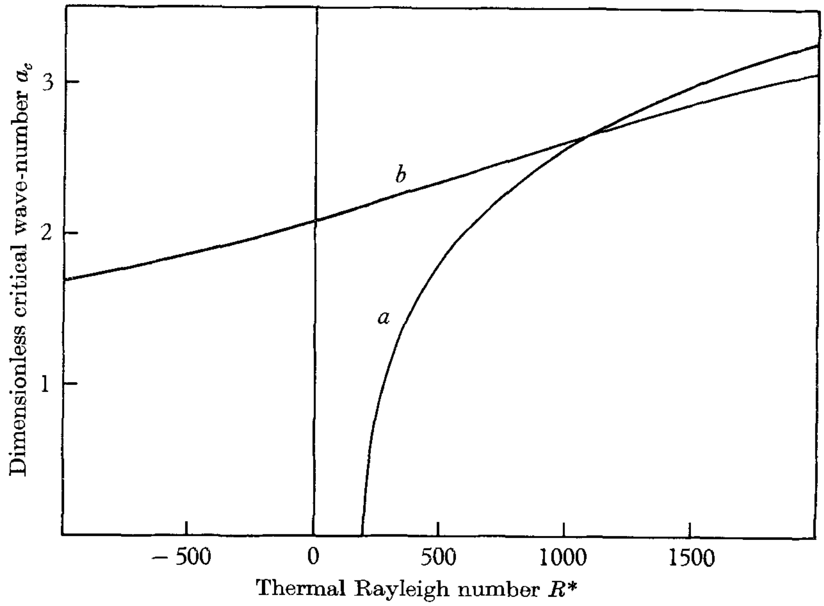

FIGURE 2. Critical wave-number corresponding to points on the neutral stability curves in figure 1.

The cut-off in case (*a*) is particularly interesting. For values of R^* smaller than 200·6 the critical value of S^* is not affected by variation in R^*. The thermal effect is then decoupled from the solute effect. The critical wave-number is zero. This means that then the layer does not split up into more than one convection cell,

K_l	K_u	L_l	L_u	R_c^*	a_c	Author
∞	∞	∞	∞	657·511	2·22	Rayleigh (1916)
∞	∞	0	∞	384·693	1·76	Present author
∞	∞	0	0	120	0	
0	∞	∞	∞	1100·657	2·68	Reid & Harris (1958)
0	∞	0	∞	816·748	2·21	Sparrow, Goldstein & Jonsson (1964)
0	∞	∞	0	669·001	2·09	
0	∞	0	0	320	0	
0	0	∞	∞	1707·765	3·12	Reid & Harris (1958)
0	0	0	∞	1295·781	2·55	Sparrow, Goldstein & Jonsson (1964)
0	0	0	0	720	0	

$K = 0$, rigid; $K = \infty$, free; $L = 0$, constant heat flux; $L = \infty$, constant temperature.

TABLE 1. Values of critical thermal Rayleigh number R_c^* (in absence of solute) and corresponding critical wave-number a_c, for various boundary conditions.

and that if the fluid is unbounded laterally, this single cell is of infinite extent In practice the presence of lateral boundaries will impose a non-zero lower bound on the horizontal wave-number, and the minimum Rayleigh number (here the solute Rayleigh number) required to cause convection will be raised. A detailed discussion of the possibility of convection at zero wave-number is presented in an appendix.

Although we have extended the stability diagram of figure 1 into the fourth quadrant, we must remember that in that quadrant there is the possibility not only of overstability with infinitesimal disturbances, but also of finite-amplitude instability of both monotonic and oscillatory types. (The corresponding stability boundary is dependent on the values of the Prandtl number and the Schmidt number for the fluid solution.) The criterion illustrated in figure 1 is thus a necessary, but in geneial not sufficient, condition for stability for states corresponding to points in the fourth quadrant. In the remainder of the plane it is thought that the curves plotted are the physically-significant stability boundaries for the cases considered.

A calculation of the stability criterion for infinitesimal oscillatory disturbances, for one set of non-ideal boundary conditions, has been reported in the author's (1966*b*) thesis. However, for qualitative comparison with the skimpy experimental results so far available, the theory for ideal boundary conditions given in § 3 should be adequate. It is noteworthy that in Shirtcliffe's experiments the oscillatory convection appeared when the temperature gradient was of order of magnitude that predicted by the linear theory. The observed period of the oscillation was likewise in agreement with this theory.

There remains the question of when the finite-amplitude stability criterion supercedes the criterion given by the linear theory. Veronis (1965) and Sani (1965) both found that for S negative and small in magnitude, infinitesimal disturbances are more unstable than finite-amplitude ones, so that our linear theory is useful in some range extending a small distance into the fourth quadrant of the (R, S)-plane. The next desirable step appears to be the extension of the finite-amplitude theory to non-ideal boundary conditions.

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Appendix

Convection at zero critical wave-number

In a strict sense the situation $b_c = 0$ is meaningless, since when $b = 0$ the equations (2.16)–(2.18) imply that W_1 is identically zero and there can be no motion. In practice, however, the presence of lateral boundaries will impose a non-zero lower bound on the horizontal wave-number, and it is sensible to consider the situation when b is small but finite. The left-hand side of the eigenvalue equation (4.14) may be expanded in powers of b. It is of the form

$$A_0 + A_2 b^2 + A_4 b^4 + \dots.$$

Thus as $b \to 0$ the eigenvalue equation reduces to $A_0 = 0$. Here A_0 does not contain R and S, but it is a function of K_l and K_u (which in our problem may have the values 0 and ∞ only) and of L_l, L_u, M_l and M_u (which must all be non-negative). It is found that A_0 vanishes identically if $L_l = L_u = 0$ or $M_l = M_u = 0$, but not otherwise under the above restrictions.

For the moment we shall concentrate on the thermal problem, when $S = 0$. The interesting case is then that when $L_l = L_u = 0$, corresponding to the condition of constant heat flux at each boundary. The eigenvalue equation now reduces to

$$A_2 + A_4 b^2 + \dots = 0, \tag{A 1}$$

where A_2 is found to be linear in R. Hence as $b \to 0$, R can be explicitly expressed in terms of the remaining parameters. When $b \to 0$,

$$\Sigma_1 \to 2\zeta_2, \quad \Sigma_3/b^2 \to 2\zeta_6, \quad \Sigma_5/b^2 \to 2\zeta_4,$$

$$\Sigma_2 \to 2\eta_2, \quad \Sigma_4/b^2 \to 2\eta_6, \quad \Sigma_6/b^2 \to 2\eta_4,$$

where
$$\zeta_s = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \eta_s = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}.$$

Then, with the aid of the approximations (to order b^2),

$$\pi b \coth \pi b = 1 + \tfrac{1}{3}\pi^2 b^2 \quad \text{and} \quad \pi b \operatorname{cosech} \pi b = 1 - \tfrac{1}{6}\pi^2 b^2,$$

and of the known values

$$\zeta_2 = \pi^2/6, \quad \zeta_4 = \pi^4/90, \quad \zeta_6 = \pi^6/945,$$

$$\eta_2 = \pi^2/12, \quad \eta_4 = 7\pi^4/720, \quad \eta_6 = 31\pi^6/30{,}240,$$

one easily obtains the following values of the Rayleigh number $R^* = \pi^4 R$.

Case (*a*). Both boundaries rigid ($K_l = 0$, $K_u = 0$).

$$R^* = \frac{(\frac{1}{4}\pi^6)(\zeta_2+\eta_2)}{(\zeta_2+\eta_2)(\zeta_6+\eta_6)-(\zeta_4+\eta_4)^2} = 720.$$

Case (*b*). One boundary rigid and the other free ($K_l = 0$, $K_u = \infty$; or $K_l = \infty$, $K_u = 0$).

$$R^* = \frac{(\frac{1}{2}\pi^6)\zeta_2}{2\zeta_2(\zeta_6+\eta_6)-(\zeta_4+\eta_4)^2} = 320.$$

Case (*c*). Both boundaries free ($K_l = \infty$, $K_u = \infty$).

$$R^* = \frac{\pi^6}{4(\zeta_6+\eta_6)} = 120.$$

The results for cases (*a*) and (*b*) confirm (to high accuracy!) the values 720·000 and 320·000 listed without comment by Sparrow, Goldstein & Jonsson (1964). The result (*c*) has not appeared in the general literature, although doubtless it has been calculated previously.† (It corresponds to a case previously considered, but with the wrong form of the boundary conditions, by Jeffreys (1926). In a later paper Jeffreys (1928) gave the corrected expressions for the boundary conditions but did not repeat his calculation.)

We can check that the above values for R^* are indeed critical Rayleigh numbers. Equation (A 1) can be written in the form

$$R = R_0 + \mu b^2 + O(b^4), \tag{A 2}$$

where the coefficient μ depends on R. For the cases considered, μ is positive when R has the value R_0. Thus R has a minimum (as b varies) at $b = 0$. In the author's experience the function $R(b)$ defined by the eigenvalue equation (4.14) has, for the present simple problem, only one minimum. The critical Rayleigh number is therefore R_0 if the layer extends to infinity in the horizontal direction. However, as mentioned above, in practice the fluid will be bounded by lateral walls, and the single convection cell will be limited in size. The corresponding value of b will not be zero, and the minimum Rayleigh number will not be R_0 but will be the somewhat higher value corresponding to this wall-limited wave-number.

In order to see whether the disturbance associated with the small critical wave-number does grow with time, we require the eigenvalue equation with non-zero time-constant σ_1. To the first order in the small quantities b^2 and σ_1 we obtain

$$Rb^2 = R_0(b^2 + r\sigma_1).$$

Thus for Rayleigh numbers near the critical value, $\sigma_1 \to 0$ as $b \to 0$ if the Prandtl number r is finite. At small but finite values of b, however, one can have σ_1 positive (though small) for slightly supercritical Rayleigh numbers. Also the perturbation velocity amplitude W_1 is small but not zero. We conclude that the disturbance is then able to grow with time (though relatively slowly), and convection can therefore occur.

† [*Note added in proof.*] The value has been published since by Hurle, Jakeman & Pike (1967).

We shall now investigate why it is that, in the thermal problem, only when the heat flux is kept fixed at each boundary is a single cell the favoured form of convection. The stabilizing factors present are viscosity and thermal diffusivity. The latter tends to make uniform the temperature of the fluid, and thus reduces the buoyancy effect. When the total heat flow across each boundary is kept constant, the perturbation heat flow out of the fluid layer is zero. While surplus heat can still diffuse back into the body of the fluid, it cannot diffuse out across the boundary. A possible thermal stabilizing effect is thus absent. Viscosity is then the dominant stabilizing factor. The favoured configuration for convection is then that for which the viscous dissipation is least. This is a single cell. On the other hand, when perturbation heat flow across a boundary is allowed, a stabilizing effect is thereby introduced. (This, incidentally, increases the critical Rayleigh number.) The contribution of this thermal stabilizing effect decreases with increasing wave-number. The viscous stabilizing effect, however, becomes greater as the wave-number increases, for then more vortices are formed. It appears that the combined stabilizing effect is a minimum for a non-zero wave-number. This critical wave-number determines the horizontal scale of the favoured convection cells.

For the thermohaline problem the coefficient μ in (A 2) depends on both R and S. For sufficiently small values of S, $\mu(R_0, S)$ is negative, but for large values of S it is positive. For the case $K_l = 0$, $K_u = \infty$, $L_l = 0$, $L_u = 0$, $\mathrm{M} = \infty$, $M_u = \infty$, one finds that the change of sign occurs when S has the value 2·060, i.e. when $S^* = 200{\cdot}6$. The critical wave-number is then zero provided that S^* is less than 200·6. This is the value quoted in the discussion (in § 5) of the analogous case in which the roles of the thermal and solute effects are reversed.

As a final example we consider a thermal problem in which surface-tension effects, as well as buoyancy effects, are important (Nield 1964). The interesting case is that when the lower boundary is rigid and the upper free surface is subject to the surface-tension condition derived by Pearson (1958). One again finds that, on the Pearson–Nield model, convection at zero wave-number occurs if and only if the heat flux across each boundary is kept constant. For such conditions the neutral stability condition reduces to

$$[2\zeta_2(\zeta_6+\eta_6)-(\zeta_4+\eta_4)^2]\,R_1+[(\zeta_2-\eta_2)\,(\zeta_4+\eta_4)]\,B_1 = 2\zeta_2(\zeta_2+\eta_2),$$

or

$$\pi^4R_1/320+\pi^2B_1/48 = 1,$$

where π^4R_1 is the Rayleigh number and π^2B_1 is the Marangoni number as defined in the author's 1964 paper for example. The surface-tension and the buoyancy effects are now perfectly coupled.

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